Furthermore, adding AO to OK we find

$$AK=\frac{(AO+AO')FO''}{AO-AO'}=\frac{(2R'+R'')r}{R''}=\frac{R''(R''+15r')^{\frac{1}{2}}(R''+15r'-1)}{R''+15r'+3},$$

and, similarly,

$$AM=\frac{(AO+AO')EQ'}{AO-AO'}=\frac{(2R'+R'')r'}{R''}=r'\surd(R''+15r').$$ Hence this

Construction.—On the indefinite straight line AX, set off

$$\begin{aligned} AB &= 2AO' = \tfrac{1}{2}R''[\surd(R''+15r')-1], \\ AD &= 2AO \ = \tfrac{1}{2}R''[\surd(R''+15r')+1], \\ BQ &= R''. \end{aligned}$$

Bisect AB in O$'$, and AD in O: then, with centers O$'$, O and Q, and with radii AO' AO and BQ, describe the circles AEB, $AHLD$ and BGD. These will obviously touch each other, two and two.

Again, on AD set off

$$AK=\frac{R''\surd(R''+15r')(R''+15r'-1)}{R''+15r'+3},$$

and $AM=r'\surd(R''+15r')$. At K, erect the perpend. KO''

$$=\frac{2R''(R''+15r'-1)}{R''+15r'+3},$$

and at M erect the perpendicular $MQ'=4r'$. Finally, with centers O$''$ and Q', and radii $FO''=R''(R''+15r'-1)\div(R''+15r'+3)$ and $EQ'=4r'$, describe the circles FIL, EIH and it is done.

[The foregoing solution, it will be seen, applies only to a special case of the question quoted at p. 143, Vol. V, as the relative position of the given circles cannot be arbitrarily taken. No general solution has been rec'd.]

NOTE ON THE DIFFERENTIATICN OF EXPCNENTIAL AND TRIGONOMETRICAL FUNCTICNS.

BY ARTEMAS MARTIN, M. A., ERIE, PA.

THE following methods may not be new, but I have not met with them in any of the many works on the calculus to which I have access.

1. Differentiate $y=a^x$.

We have, by the Exponential Theorem,

$$y=a^x=1+x\log a+\frac{x^2(\log a)^2}{1.2}+\frac{x^3(\log a)^3}{1.2.3}+\text{ etc};$$

$$\therefore dy = d(a^x) = (\log a)dx + x(\log a)^2dx + \frac{x^2(\log a)^3dx}{1\,.\,2} + \text{ etc.},$$

$$= \log a\left[1 + x\log a + \frac{x^2(\log a)^2}{1\,.\,2} + \frac{x^3(\log a)^3}{1\,.\,2\,.\,3} + \text{ etc.}\right]dx,$$

$$= a^x \log a\,dx.$$

2. Differentiate $y = \sin x$.

We have, by Analytical Trigonometry,

$$y = \sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} + \text{ etc.},$$

$$\cos x = 1 - \frac{x^2}{1\,.\,2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \text{ etc.};$$

$$\therefore \quad dy = d(\sin x) = dx \; - \frac{x^2dx}{1\,.\,2} + \frac{x^4dx}{1.2.3.4} - \frac{x^6dx}{1.2.3.4.5.6} + \text{ etc.},$$

$$= \left[1 - \frac{x^2}{1\,.\,2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \text{ etc.}\right]dx,$$

$$= \cos x\,dx.$$

Evaluation of the Eulerian Integral $u = \int_0^{\frac{\pi}{2}} \log(\sin x)\,dx$, By Prof. E. J. Edmunds, New Orleans, La.—We have

$$u = \text{limit } \frac{\pi}{2n}\left[\log\sin\left(\frac{1}{n}.\frac{\pi}{2}\right) + \log\sin\left(\frac{2}{n}.\frac{\pi}{2}\right) - \ldots + \log\sin\left(\frac{n-1}{n}.\frac{\pi}{2}\right)\right];$$

hence,

$$u = \lim.\frac{\pi}{2n}.\log\left[\sin\left(\frac{1}{n}.\frac{\pi}{2}\right)\sin\left(\frac{2}{n}.\frac{\pi}{2}\right)\ldots\ldots\sin\left(\frac{n-1}{n}.\frac{\pi}{2}\right)\right],$$

n increasing to infinity.

But by Cotes' Theorem we have

$$\frac{z^{2n}-1}{z^n-1} = \left(z^2 - 2z\cos\frac{\pi}{n} + 1\right)\left(z^2 - 2z\cos\frac{2\pi}{n} + 1\right)\ldots\left(z^2 - 2z\cos\frac{(n-1)\pi}{n} + 1\right).$$

For $z = 1$, this equation becomes,

$$n = 2^{2(n-1)}\sin^2\left(\frac{1}{n}.\frac{\pi}{2}\right)\sin^2\left(\frac{2}{n}.\frac{\pi}{2}\right)\ldots\sin^2\left(\frac{n-1}{n}.\frac{\pi}{2}\right); \text{ therefore}$$

$$\frac{\log n - 2(n-1)\log 2}{2} = \log\sin\left(\frac{1}{n}.\frac{\pi}{2}\right)\ldots\log\sin\left(\frac{n-1}{n}.\frac{\pi}{2}\right).$$

Multiplying both members by $\pi \div 2n$ and passing to limits we finally obtain

$$u = \tfrac{1}{2}\pi.\log\tfrac{1}{2},$$

which is the value of the integral sought.